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LETTER TO THE EDITOR

Algebraic Bethe ansatz for the one-dimensional Bariev chain

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Abstract. The algebraic Bethe ansatz is formulated for the one-dimensional Bariev chain. As a consequence, the eigenvectors, eigenvalues and Bariev's Bethe equations are rederived in a systematic algebraic way.

The discovery of high-T_c superconductivity has greatly stimulated studies of various electron lattice models in one dimension [1–8], which are exactly soluble using the coordinate space Bethe ansatz method [9]. Notably, many physical properties of these models, such as the one-dimensional (1D) Hubbard model [5], the supersymmetric t-J model [2] and the 1D Bariev model [6, 10], have been inferred from the Bethe equations. However, it still seems to be interesting to investigate these models within the framework of the quantum inverse scattering method (QISM) [9, 11, 12]. This would provide us with more information on their algebraic structure. At present, the QISM has been successfully applied to the 1D Hubbard model [7, 13, 14] and to the supersymmetric t-J model [8, 15]. For the 1D Bariev model, the integrabilty is also established by showing the quantum Yang–Baxter relation [16]. Therefore, it is desirable to derive Bariev's Bethe equations using the algebraic Bethe ansatz technique, as done by Ramos and Martins [14] for the 1D Hubbard model.

This letter addresses this question. Our approach is based on the algebraic Bethe ansatz technique developed by Tarasov [17], which has recently been extended to apply to the diagonalization of the 1D Hubbard model [14]. It is found that a novel set of commutation relations between various elements of the monodromy matrix may be inferred from the quantum Yang–Baxter relation, which uncover a hidden six-vertex symmetry underlying the integrability of the model. As a consequence, the eigenvectors, eigenvalues and Bariev's Bethe equations are rederived in a systematic algebraic way.

The Hamiltonian of the 1D Bariev periodic chain may be written in the form [6]

$$H=\sum_{j=1}^{N}[(c_{j\uparrow}^{\dagger}c_{j+1\uparrow}+c_{j+1\uparrow}^{\dagger}c_{j\uparrow})\exp(\eta n_{j+1\downarrow})+(c_{j\downarrow}^{\dagger}c_{j+1\downarrow}+c_{j+1\downarrow}^{\dagger}c_{j\downarrow})\exp(\eta n_{j\uparrow})]. \tag{1}$$

Here $c_{j\alpha}^{+}$ and $c_{j\alpha}$ are, respectively, the creation and annihilation operators of fermions with spin $\alpha(=\uparrow$ or $\downarrow)$ at a site j and $n_{j\alpha}$ is the density operator.

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As was shown in [16], the Lax operator $L_j(\lambda)$ takes the form

$$L_j(\lambda) = \tilde{L}_j(\lambda)\tilde{\tilde{L}}_j(\lambda) \tag{2}$$

with

$$\tilde{L}_j(\lambda) = \begin{pmatrix} \lambda\exp(\eta) + (\mathrm{i} - \lambda\exp(\eta))n_{j\uparrow} & 0 & -\mathrm{i}\sqrt{1+\exp(2\eta)\lambda^2}c_{j\uparrow} & 0 \\ 0 & \lambda + (\mathrm{i}-\lambda)n_{j\uparrow} & 0 & \mathrm{i}\sqrt{1+\lambda^2}c_{j\uparrow} \\ \sqrt{1+\exp(2\eta)\lambda^2}c^\dagger_{j\uparrow} & 0 & 1-(1+\mathrm{i}\lambda\exp(\eta))n_{j\uparrow} & 0 \\ 0 & -\sqrt{1+\lambda^2}c^\dagger_{j\uparrow} & 0 & 1-(1+\mathrm{i}\lambda)n_{j\uparrow} \end{pmatrix} \tag{3}$$

and

$$\tilde{\tilde{L}}_j(\lambda) = \begin{pmatrix} \lambda\exp(\eta) + (\mathrm{i} - \lambda\exp(\eta))n_{j\downarrow} & \sqrt{1+\exp(2\eta)\lambda^2}c_{j\downarrow} & 0 & 0 \\ -\mathrm{i}\sqrt{1+\exp(2\eta)\lambda^2}c^\dagger_{j\downarrow} & 1-(1+\mathrm{i}\lambda\exp(\eta))n_{j\downarrow} & 0 & 0 \\ 0 & 0 & \lambda+(\mathrm{i}-\lambda)n_{j\downarrow} & \sqrt{1+\lambda^2}c_{j\downarrow} \\ 0 & 0 & -\mathrm{i}\sqrt{1+\lambda^2}c^\dagger_{j\downarrow} & 1-(1+\mathrm{i}\lambda)n_{j\downarrow} \end{pmatrix} \tag{4}$$

where λ denotes the spectral parameter. Indeed, one may check that the quantum R-matrix satisfying

$$R(\lambda,\mu)L_j(\lambda)\otimes_s L_j(\mu) = L_j(\mu)\otimes_s L_j(\lambda)R(\lambda,\mu) \tag{5}$$

does exist. Here $\otimes_s$ denotes the supertensor product defined by

$$(A\otimes_s B)_{ik,jl} = (-1)^{[P(i)+P(j)]P(k)}A_{ij}B_{kl} \tag{6}$$

where $P(1) = P(4) = 0, P(2) = P(3) = 1$. The explicit form of the R-matrix may be found in [16]. Since our aim is to diagonalize the model (1) using the algebraic Bethe ansatz approach, it is crucial to write down some fundamental commutation relations between various elements of the monodromy matrix $T(\lambda)$, which is defined as $T(\lambda) = L_N(\lambda)\dots L_1(\lambda)$. From (5) it follows that

$$R(\lambda,\mu)T(\lambda)\otimes_s T(\mu) = T(\mu)\otimes_s T(\lambda)R(\lambda,\mu). \tag{7}$$

For our purpose, it is convenient to assume that

$$T(\lambda) = \begin{pmatrix} A(\lambda) & \boldsymbol{C}^*(\lambda) & F^*(\lambda) \\ \boldsymbol{B}^*(\lambda) & \hat{A}(\lambda) & \boldsymbol{C}(\lambda) \\ F(\lambda) & \boldsymbol{B}(\lambda) & D(\lambda) \end{pmatrix} \tag{8}$$

where the operators $\boldsymbol{B}(\lambda), \boldsymbol{B}^*(\lambda)$ and $\boldsymbol{C}(\lambda)\boldsymbol{C}^*(\lambda)$ are two component vectors with dimensions $1\times 2(2\times 1)$ and $2\times 1(1\times 2)$, respectively. The operator is a 2×2 matrix and the other remaining ones are scalars. Then, it follows from (7) that

$$\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}(\mu) = \boldsymbol{B}(\mu)\otimes\boldsymbol{B}(\lambda)\hat{r}(\lambda,\mu) - \mathrm{i}\frac{\sqrt{(1+\exp(2\eta)\lambda^2)(1+\mu^2)}}{\exp(2\eta)\lambda-\mu} \times[F(\lambda)D(\mu) - F(\mu)D(\lambda)]\boldsymbol{\xi} \tag{9}$$

$$D(\lambda)\boldsymbol{B}(\mu) = \mathrm{i}\frac{1+\lambda\mu}{\lambda-\mu}\boldsymbol{B}(\mu)D(\lambda) - \mathrm{i}\frac{\sqrt{(1+\lambda^2)(1+\mu^2)}}{\lambda-\mu}\boldsymbol{B}(\lambda)D(\mu) \tag{10}$$

$$\begin{aligned} A(\lambda)\boldsymbol{B}(\mu) &= \mathrm{i}\frac{\exp(\eta)(1+\lambda\mu)}{\exp(2\eta)\lambda-\mu}\boldsymbol{B}(\mu)A(\lambda) \\ &\quad + \frac{\sqrt{(1+\lambda^2)(1+\exp(2\eta)\lambda^2)(1+\mu^2)(1+\exp(2\eta)\mu^2)}}{(\lambda-\mu)(\exp(2\eta)\lambda-\mu)}F(\lambda)\boldsymbol{C}^*(\mu) \\ &\quad - \frac{\sqrt{(1+\exp(2\eta)\lambda^2)(1+\exp(2\eta)\mu^2)}(1+\lambda\mu)}{(\lambda-\mu)(\exp(2\eta)\lambda-\mu)}F(\mu)\boldsymbol{C}^*(\lambda) \\ &\quad - \mathrm{i}\frac{\sqrt{(1+\exp(2\eta)\lambda^2)(1+\mu^2)}}{\exp(2\eta)\lambda-\mu}\boldsymbol{\xi}(\boldsymbol{B}^*(\lambda)\hat{A}(\mu)) \end{aligned} \tag{11}$$

and

$$\hat{A}(\lambda)\otimes \boldsymbol{B}(\mu) = \mathrm{i}\frac{1+\lambda\mu}{\lambda-\mu}[\boldsymbol{B}(\mu)\otimes\hat{A}(\lambda)]\hat{r}(\lambda,\mu) - \mathrm{i}\frac{\sqrt{(1+\lambda^2)(1+\mu^2)}}{\lambda-\mu}\boldsymbol{B}(\lambda)\otimes\hat{A}(\mu)$$
$$-\mathrm{i}\frac{\sqrt{(1+\exp(2\eta)\lambda^2)(1+\mu^2)}}{\exp(2\eta)\lambda-\mu}\Bigg[\boldsymbol{B}^*(\lambda)D(\mu) - \mathrm{i}\frac{\sqrt{(1+\lambda^2)(1+\mu^2)}}{\lambda-\mu}F(\lambda)\boldsymbol{C}(\mu)$$
$$+\mathrm{i}\frac{\sqrt{(1+\lambda^2)(1+\exp(2\eta)\mu^2)}(1+\lambda\mu)}{\sqrt{(1+\exp(2\eta)\lambda^2)(1+\mu^2)}(\lambda-\mu)}F(\mu)\boldsymbol{C}(\lambda)\Bigg]\otimes\boldsymbol{\xi} \tag{12}$$

where

$$\boldsymbol{\xi} = (0, \exp(\eta), -1, 0) \tag{13}$$

$$\hat{r}(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(\exp(2\eta)-1)\mu}{\exp(2\eta)\lambda-\mu} & \frac{\exp(\eta)(\lambda-\mu)}{\exp(2\eta)\lambda-\mu} & 0 \\ 0 & \frac{\exp(\eta)(\lambda-\mu)}{\exp(2\eta)\lambda-\mu} & \frac{(\exp(2\eta)-1)\lambda}{\exp(2\eta)\lambda-\mu} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

It should be noted that the $\hat{r}(\lambda,\mu)\equiv\hat{r}(\lambda/\mu)$ is nothing but the R-matrix of the symmetric six-vertex model in the so-called homogeneous gauge used by Jimbo [18], which differs from the usual symmetric form by a gauge transformation [19]. Here it seems appropriate to mention that a similar situation also occurs in the 1D Hubbard model, which was first discovered by Ramos and Martins [14]. It should be stressed that such hidden symmetry gives an algebraic explanation for the fact that the bare two-body scattering matrix calculated in the coordinate space Bethe ansatz approach appears in the six-vertex form.

Next we proceed to establish the Bethe eigenvectors. First note that the monodromy matrix $T(\lambda)$ is a lower-triangular matrix when acting on the pseudovacuum defined by $a_j|0\rangle = 0$, $j=1,\ldots,N$. Explicitly, we have

$$A(\lambda)|0\rangle = (\lambda\exp(2\eta))^N|0\rangle \qquad D(\lambda) = |0\rangle \qquad \hat{A}_{aa}|0\rangle = \lambda^N|0\rangle \qquad (a=1,2) \tag{15}$$

and

$$\boldsymbol{C}(\lambda)|0\rangle = \boldsymbol{C}^*(\lambda)|0\rangle = F^*(\lambda)|0\rangle = 0 \qquad \hat{A}_{ab}|0\rangle = 0 \qquad (a\neq b = 1,2). \tag{16}$$

This implies that one may view the operators $\boldsymbol{B}(\lambda), \boldsymbol{B}^*(\lambda)$ and $F(\lambda)$ as the creation operators on the pseudovacuum $|0\rangle$. Keeping this fact in mind and noting the commutation relations (9), one may conclude that the one-particle state $|\psi_1(\lambda_1)\rangle$ takes the form

$$|\psi_1(\lambda_1)\rangle = \boldsymbol{B}(\lambda_1)\boldsymbol{F}_1|0\rangle \tag{17}$$

whereas the two-particle state $|\psi_2(\lambda_1,\lambda_2)\rangle$ is given by

$$|\psi_2(\lambda_1,\lambda_2)\rangle = \Bigg[\boldsymbol{B}(\lambda_1)\otimes\boldsymbol{B}(\lambda_2) + \mathrm{i}\frac{\sqrt{(1+\exp(2\eta)\lambda_1^2)(1+\lambda_2^2)}}{\exp(2\eta)\lambda_1-\lambda_2}$$
$$\times F(\lambda_1)(\boldsymbol{\xi}\otimes\boldsymbol{\psi}_0)D(\lambda_2)\Bigg]\boldsymbol{F}_2|0\rangle. \tag{18}$$

Here and hereafter, $\boldsymbol{F}_n$ denotes a constant vector with dimension $2^n\times 1$. In general, the n-particle state may be constructed in a recursive way as follows

$$|\psi_n(\lambda_1,\ldots,\lambda_n)\rangle = \boldsymbol{\psi}_n(\lambda_1,\ldots,\lambda_n)\boldsymbol{F}_n|0\rangle \tag{19}$$

where

$$\begin{aligned}\boldsymbol{\psi}_n(\lambda_1,\ldots,\lambda_n) &= \boldsymbol{B}(\lambda_1)\otimes\boldsymbol{\psi}_{n-1}(\lambda_2,\ldots,\lambda_n)\\ &\quad +F(\lambda_1)\boldsymbol{\xi}\otimes\sum_{j=2}^{n}\mathrm{i}\frac{\sqrt{(1+\exp(2\eta)\lambda_1^2)(1+\lambda_j^2)}}{\exp(2\eta)\lambda_1-\lambda_j}\prod_{\substack{k=2\\k\neq j}}^{n}\frac{\mathrm{i}(1+\lambda_k\lambda_j)}{\lambda_k-\lambda_j}\\ &\quad\times\boldsymbol{\psi}_{n-2}(\lambda_2,\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_n)\prod_{k=2}^{j-1}\hat{r}_{k,k+1}(\lambda_k,\lambda_j).\end{aligned}\tag{20}$$

Taking into account these results and noting the commutation relations (10)–(12), we conclude that the eigenvalue $\Lambda(\lambda,\{\lambda_j\})$ of the transfer matrix of the model, which is defined as the supertrace of the monodromy matrix $T(\lambda)$, takes the form

$$\begin{aligned}\Lambda(\lambda,\{\lambda_j\}) &= \prod_{j=1}^{n}\frac{\mathrm{i}(1+\lambda\lambda_j)}{\lambda-\lambda_j}+(\exp(\eta)\lambda)^{2N}\prod_{j=1}^{n}\frac{\mathrm{i}(1+\lambda\lambda_j)}{\exp(\eta)\lambda-\exp(-\eta)\lambda_j}\\ &\quad -\lambda^N\prod_{j=1}^{n}\frac{\mathrm{i}(1+\lambda\lambda_j)}{\lambda-\lambda_j}\Lambda^{(1)}(\lambda,\{\lambda_j\},\{\mu_\alpha\})\end{aligned}\tag{21}$$

where $\Lambda^{(1)}(\lambda,\{\lambda_j\},\{\mu_\alpha\})$ is the corresponding eigenvalue of an auxiliary inhomogeneous transfer matrix, which is inherited from the hidden six-vertex symmetry mentioned above. The final result is as follows,

$$\begin{aligned}\Lambda(\lambda,\{\lambda_j\},\{\mu_\alpha\}) &= \prod_{j=1}^{n}\frac{\mathrm{i}(1+\lambda\lambda_j)}{\lambda-\lambda_j}+(\exp(\eta)\lambda)^{2N}\prod_{j=1}^{n}\frac{\mathrm{i}(1+\lambda\lambda_j)}{\exp(\eta)\lambda-\exp(-\eta)\lambda_j}\\ &\quad -\lambda^N\prod_{j=1}^{n}\frac{\mathrm{i}(1+\lambda\lambda_j)}{\lambda-\lambda_j}\prod_{\alpha=1}^{m}\frac{\exp(-\eta)\lambda-\exp(\eta)\mu_\alpha}{\lambda-\mu_\alpha}\\ &\quad -\lambda^N\prod_{j=1}^{n}\frac{\mathrm{i}(1+\lambda\lambda_j)}{\exp(\eta)\lambda-\exp(-\eta)\lambda_j}\prod_{\alpha=1}^{m}\frac{\exp(\eta)\lambda-\exp(-\eta)\mu_\alpha}{\lambda-\mu_\alpha}\end{aligned}\tag{22}$$

provided that the parameters $\{\lambda_j\}$ and $\{\mu_\alpha\}$ satisfy the Bethe equations,

$$\lambda_j^{-N}=\prod_{\alpha=1}^{m}\frac{\exp(-\eta)\lambda_j-\exp(\eta)\mu_\alpha}{\lambda_j-\mu_\alpha}\qquad j=1,2,\ldots,n\tag{23}$$

$$\prod_{j=1}^{n}\frac{\exp(\eta)\mu_\beta-\exp(-\eta)\lambda_j}{\mu_\beta-\lambda_j}=-\prod_{\alpha=1}^{m}\frac{\exp(\eta)\mu_\beta-\exp(-\eta)\mu_\alpha}{\exp(-\eta)\mu_\beta-\exp(\eta\mu_\alpha)}\qquad \alpha,\beta=1,2,\ldots,m.\tag{24}$$

After a redefinition of the parameters $\{\lambda_j\}$ and $\{\mu_\alpha\}$, we may check that our conclusion is consistent with that obtained by Bariev [6] using the coordinate space Bethe ansatz. Also, the eigenvalue E of the Hamiltonian (1) is

$$E=2\sum_{j=1}^{n}(\lambda_j+\lambda_j^{-1}).\tag{25}$$

In conclusion, we have presented the algebraic Bethe ansatz for the 1D Bariev chain. This allows us to construct the eigenvectors, eigenvalues and the Bethe equations in a systematic algebraic way. Our result is consistent with that obtained by Bariev [6] using the coordinate space Bethe ansatz method and may be useful in understanding the completeness of the Bethe eigenvectors.

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